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# ON THE STABILITY OF THE QUADRATIC-ADDITIVE FUNCTIONAL EQUATION IN RANDOM NORMED SPACES VIA FIXED POINT METHOD

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Abstract. In this paper, we prove the stability in random normed spaces via fixed point method for the functional equation

$$f(x+y+z+w)+2f(x)+2f(y)+2f(z)+2f(w)-f(x+y)$$
$$-f(x+z)-f(x+w)-f(y+z)-f(y+w)-f(z+w)=0.$$

## 1. Introduction

In 1940, S. M. Ulam [26] raised a question concerning the stability of homomorphisms: Given a group $G_1$, a metric group $G_2$ with the metric $d(\cdot,\cdot)$, and a positive number $\varepsilon$, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G_1$ then there exists a homomorphism $F : G_1 \to G_2$ with

$$d(f(x), F(x)) < \varepsilon$$

for all $x \in G_1$? As mentioned above, when this problem has a solution, we say that the homomorphisms from $G_1$ to $G_2$ are stable. In 1941, D. H. Hyers [5] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption that $G_1$ and $G_2$ are Banach spaces. Hyers' result was generalized by T. Aoki [1] for additive mappings and Th. M. Rassias [22] for linear mappings by considering the stability problem with unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see

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[3], [4], [7]-[18].

Recall, almost all subsequent proofs in this very active area have used Hyers' method, called *a direct method.* Namely, the function $F$, which is the solution of a functional equation, is explicitly constructed, starting from the given function $f$, by the formulae $F(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ or $F(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$. In 2003, V. Radu [21] observed that the existence of the solution $F$ of a functional equation and the estimation of the difference with the given function $f$ can be obtained from the fixed point alternative. In 2008, D. Mihet and V. Radu [20] applied this method to prove the stability theorems of *the Cauchy functional equation:*

$$f(x+y) - f(x) - f(y) = 0 \tag{1.1}$$

in random normed spaces. We call solutions of (1.1) *additive mappings.*

In 2004, Chang et al [2] established the general solution and investigated the stability of *the quadratic-additive functional equation:*

$$\begin{aligned} &f(x+y+z+w) + 2f(x) + 2f(y) + 2f(z) + 2f(w) - f(x+y) \\ &-f(x+z) - f(x+w) - f(y+z) - f(y+w) - f(z+w) = 0 \end{aligned} \tag{1.2}$$

by using a direct method. Now, we consider the functional equation:

$$\begin{aligned} f(x+y+z+w) &+ 2f(x) + 2f(y) + 2f(z) + 2f(w) - f(x+y) \\ &-f(x+z) - f(x+w) - f(y+z) - f(y+w) - f(z+w) \\ &-3f(0) = 0 \end{aligned} \tag{1.3}$$

which is called *the general quadratic functional equation.* In this paper, using the fixed point method, we prove the stability for the functional equation (1.2) and the general quadratic functional equation (1.3) in random normed spaces. It is easy to see that the mappings $f(x) = ax^2 + bx$ and $f(x) = ax^2 + bx + c$ are solutions of the functional equation (1.2) and (1.3), respectively. Every solution of the quadratic-additive functional equation (1.2) and the general quadratic functional equation (1.3) are said to be *a quadratic-additive mapping* and *a general quadratic mapping,* respectively.

## 2. Preliminaries

In this section, we state the usual terminology, notations and conventions of the theory of random normed spaces, as in [24,25]. Firstly, the space of all probability distribution functions is denoted by

$$\begin{aligned}\Delta^+ \quad := \quad & \{F : \mathbb{R}\cup\{-\infty,\infty\} \to [0,1] \big| F \text{ is left-continuous} \\ & \text{and nondecreasing on } \mathbb{R}, \text{ where } F(0)=0 \text{ and } F(+\infty)=1\}.\end{aligned}$$

And let the subset $D^+ \subseteq \Delta^+$ be the set $D^+ := \{F \in \Delta^+ | l^- F(+\infty) = 1\}$, where $l^- f(x)$ denotes the left limit of the function $f$ at the point $x$. The space $\Delta^+$ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for $\Delta^+$ in this order is the distribution function $\varepsilon_0 : \mathbb{R}\cup\{0\} \to [0,\infty)$ given by

$$\varepsilon_0(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

Definition 2.1. ([24]) A mapping $\tau : [0,1]\times[0,1] \to [0,1]$ is called *a continuous triangular norm* (briefly, *a continuous t-norm*) if $\tau$ satisfies the following conditions:
(a) $\tau$ is commutative and associative;
(b) $\tau$ is continuous;
(c) $\tau(a,1) = a$ for all $a \in [0,1]$;
(d) $\tau(a,b) \leq \tau(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous $t$-norms are $\tau_P(a,b) = ab$, $\tau_M(a,b) = \min(a,b)$ and $\tau_L(a,b) = \max(a+b-1,0)$.

Definition 2.2. ([25]) *A random normed space* (briefly, *RN-space*) is a triple $(X,\Lambda,\tau)$, where $X$ is a vector space, $\tau$ is a continuous $t$-norm, and $\Lambda$ is a mapping from $X$ into $D^+$ such that the following conditions hold:
(RN1) $\Lambda_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$,
(RN2) $\Lambda_{\alpha x}(t) = \Lambda_x(t/|\alpha|)$ for all $x$ in $X$, $\alpha \neq 0$ and all $t \geq 0$,
(RN3) $\Lambda_{x+y}(t+s) \geq \tau(\Lambda_x(t), \Lambda_y(s))$ for all $x,y \in X$ and all $t,s \geq 0$.

If $(X, \|\cdot\|)$ is a normed space, we can define a mapping $\Lambda : X \to D^+$ by

$$\Lambda_x(t) = \frac{t}{t+\|x\|}$$

for all $x \in X$ and $t > 0$. Then $(X,\Lambda,\tau_M)$ is a random normed space, which is called *the induced random normed space.*

Definition 2.3. Let $(X,\Lambda,\tau)$ be an $RN$-space.
(i) A sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ if, for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer $N$ such that $\Lambda_{x_n - x}(t) > 1 - \varepsilon$ whenever $n \geq N$.

(ii) A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if, for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer $N$ such that $\Lambda_{x_n - x_m}(t) > 1 - \varepsilon$ whenever $n \geq m \geq N$.
(iii) An RN-space $(X, \Lambda, \tau)$ is said to be complete if and only if every Cauchy sequence in $X$ is convergent to a point in $X$.

THEOREM 2.4. ([24]) *If $(X, \Lambda, \tau)$ is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \Lambda_{x_n}(t) = \Lambda_x(t)$.*

## 3. On the stability of the quadratic-additive functional equation in RN-spaces

We recall the fundamental result in the fixed point theory.

THEOREM 3.1. ([19] or [23]) *Suppose that a complete generalized metric space $(X, d)$, which means that the metric $d$ may assume infinite values, and a strictly contractive mapping $J : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = +\infty, \ \forall n \in \mathbb{N} \cup \{0\},$$

*or there exists a nonnegative integer $k$ such that:*
(1) *$d(J^n x, J^{n+1} x) < +\infty$ for all $n \geq k$;*
(2) *the sequence $\{J^n x\}$ is convergent to a fixed point $y^*$ of $J$;*
(3) *$y^*$ is the unique fixed point of $J$ in $Y := \{y \in X, d(J^k x, y) < +\infty\}$;*
(4) *$d(y, y^*) \leq (1/(1-L)) d(y, Jy)$ for all $y \in Y$.*

Let $X$ and $Y$ be vector spaces. We use the following abbreviations for a given mapping $f : X \to Y$ by

$$\begin{aligned}
&Df(x,y,z,w)\\
&\quad := f(x+y+z+w) + 2f(x) + 2f(y) + 2f(z) + 2f(w)\\
&\qquad -f(x+y) - f(x+z) - f(x+w) - f(y+z) - f(y+w)\\
&\qquad -f(z+w),\\
&D'f(x,y,z,w)\\
&\quad := f(x+y+z+w) + 2f(x) + 2f(y) + 2f(z) + 2f(w)\\
&\qquad -f(x+y) - f(x+z) - f(x+w) - f(y+z) - f(y+w)\\
&\qquad -f(z+w) - 3f(0)
\end{aligned}$$

for all $x, y, z, w \in X$.

LEMMA 3.2. ([6]) *If $f : X \to Y$ is a mapping such that $Df(x,y,z,w) = 0$ for all $x,y,z,w \in X\backslash\{0\}$, then $f$ is a quadratic-additive mapping.*

Now we will establish the stability for the functional equation (1.2) in random normed spaces via fixed point method.

THEOREM 3.3. *Let $X$ be a linear space, $(Z,\Lambda',\tau_M)$ be an RN-space, $(Y,\Lambda,\tau_M)$ be a complete RN-space and $f : X \to Y$ be a mapping for which there is $\varphi : (X\backslash\{0\})^4 \to Z$ such that*

$$\Lambda_{Df(x,y,z,w)}(t) \geq \Lambda'_{\varphi(x,y,z,w)}(t) \tag{3.1}$$

*for all $x,y,z,w \in X\backslash\{0\}$ and $t > 0$. If for all $x,y,z,w \in X\backslash\{0\}$ and $t > 0$ $\varphi$ satisfies one of the following conditions:*
(i) $\Lambda'_{\alpha\varphi(x,y,z,w)}(t) \leq \Lambda'_{\varphi(2x,2y,2z,2w)}(t) \leq \Lambda'_{\alpha'\varphi(x,y,z,w)}(t)$ *for some* $1 < \alpha' \leq \alpha < 2$,
(ii) $\Lambda'_{\alpha\varphi(x,y,z,w)}(t) \leq \Lambda'_{\varphi(2x,2y,2z,2w)}(t)$ *for some* $0 < \alpha < 1$,
(iii) $\Lambda'_{\varphi(2x,2y,2z,2w)}(t) \leq \Lambda'_{\alpha\varphi(x,y,z,w)}(t)$ *for some* $4 < \alpha$
*then there exists a unique quadratic-additive mapping $F : X \to Y$ such that*

$$\Lambda_{f(x)-F(x)}(t) \geq \begin{cases} M(x, 2(2-\alpha)t) & \text{if } \varphi \text{ satisfies (i) or (ii),} \\ M(x, 2(\alpha-4)t) & \text{if } \varphi \text{ satisfies (iii)} \end{cases} \tag{3.2}$$

*for all $x \in X$ and $t > 0$, where*

$$M(x,t) := \tau_M\big\{\Lambda'_{\varphi(x,x,x,-x)}(t), \Lambda'_{\varphi(-x,-x,-x,x)}(t)\big\}.$$

*Moreover $\Lambda'_{\varphi(x,y,z,w)}$ is continuous in x,y,z,w under the condition* (ii), *then $f$ is a quadratic-additive mapping.*

*Proof.* Notice that

$$\begin{aligned} f(0) &= \frac{1}{3}\big(Df(x,x,x,x) + Df(-x,-x,-x,-x) \\ &\quad + Df(2x,2x,-2x,-2x) - 2Df(x,x,-x,-x)\big) \end{aligned} \tag{3.3}$$

for any $x \in X\backslash\{0\}$. We will prove the theorem in three cases, $\varphi$ satisfies one of the conditions (i), (ii) or (iii).
**Case 1.** Assume that $\varphi$ satisfies the condition (i). Choose a fixed

$x \in X\backslash\{0\}$, then it follows from (3.1), (3.3), (RN2), and (RN3) that

$$\begin{aligned}
\Lambda_{f(0)}(5t) &\geq \tau_M\Big\{\Lambda_{\frac{1}{3}Df(\frac{x}{2^n},\frac{x}{2^n},\frac{x}{2^n},\frac{x}{2^n})}(t), \Lambda_{\frac{1}{3}Df(-\frac{x}{2^n},-\frac{x}{2^n},-\frac{x}{2^n},-\frac{x}{2^n})}(t),\\
&\quad \Lambda_{\frac{1}{3}Df(\frac{2x}{2^n},\frac{2x}{2^n},-\frac{2x}{2^n},-\frac{2x}{2^n})}(t), \Lambda_{\frac{2}{3}Df(\frac{x}{2^n},\frac{x}{2^n},-\frac{x}{2^n},-\frac{x}{2^n})}(2t)\Big\}\\
&= \tau_M\Big\{\Lambda_{Df(\frac{x}{2^n},\frac{x}{2^n},\frac{x}{2^n},\frac{x}{2^n})}(3t), \Lambda_{Df(-\frac{x}{2^n},-\frac{x}{2^n},-\frac{x}{2^n},-\frac{x}{2^n})}(3t),\\
&\quad \Lambda_{Df(\frac{2x}{2^n},\frac{2x}{2^n},-\frac{2x}{2^n},-\frac{2x}{2^n})}(3t), \Lambda_{Df(\frac{x}{2^n},\frac{x}{2^n},-\frac{x}{2^n},-\frac{x}{2^n})}(3t)\Big\}\\
&\geq \tau_M\Big\{\Lambda'_{\varphi(\frac{x}{2^n},\frac{x}{2^n},\frac{x}{2^n},\frac{x}{2^n})}(3t), \Lambda'_{\varphi(-\frac{x}{2^n},-\frac{x}{2^n},-\frac{x}{2^n},-\frac{x}{2^n})}(3t),\\
&\quad \Lambda'_{\varphi(\frac{2x}{2^n},\frac{2x}{2^n},-\frac{2x}{2^n},-\frac{2x}{2^n})}(3t), \Lambda'_{\varphi(\frac{x}{2^n},\frac{x}{2^n},-\frac{x}{2^n},-\frac{x}{2^n})}(3t)\Big\}\\
&\geq \tau_M\Big\{\Lambda'_{\frac{1}{\alpha'^n}\varphi(x,x,x,x)}(3t), \Lambda'_{\frac{1}{\alpha'^n}\varphi(-x,-x,-x,-x)}(3t),\\
&\quad \Lambda'_{\frac{1}{\alpha'^n}\varphi(2x,2x,-2x,-2x}(3t), \Lambda'_{\frac{1}{\alpha'^n}\varphi(x,x,-x,-x)}(3t)\Big\}\\
&\geq \tau_M\Big\{\Lambda'_{\varphi(x,x,x,x)}\left(3\alpha'^n t\right), \Lambda'_{\varphi(-x,-x,-x,-x)}\left(3\alpha'^n t\right),\\
&\quad \Lambda'_{\varphi(2x,2x,-2x,-2x}(3\alpha'^n t), \Lambda'_{\varphi(x,x,-x,-x)}\left(3\alpha'^n t\right)\Big\}
\end{aligned}$$

for all $t > 0$ and $n \in \mathbb{N}$. Since all terms on the right hand side of the above inequality tend to 1 as $n \to \infty$, we have $f(0) = 0$ by (RN1). Let $S$ be the set of all functions $g : X \to Y$ with $g(0) = 0$ and introduce a generalized metric on $S$ by

$$d(g,h) := \inf\left\{u \in \mathbb{R}^+ \middle| \Lambda_{g(x)-h(x)}(ut) \geq M(x,t) \text{ for all } x \in X \setminus \{0\}\right\}.$$

Consider the mapping $J : S \to S$ defined by

$$Jf(x) := \frac{f(2x) - f(-2x)}{4} + \frac{f(2x) + f(-2x)}{8},$$

then we have

$$J^n f(x) = \frac{1}{2}\left(4^{-n}\left(f(2^n x) + f(-2^n x)\right) + 2^{-n}\left(f(2^n x) - f(-2^n x)\right)\right)$$

for all $x \in X$ and $n \in \mathbb{N}$. Let $f, g \in S$ and let $u \in [0, \infty]$ be an arbitrary constant with $d(g, f) \leq u$. From the definition of $d$, (RN2), and (RN3), for the given $0 < \alpha < 2$ we have

$$\begin{aligned}
&\Lambda_{Jg(x)-Jf(x)}\left(\frac{\alpha u}{2}t\right) = \Lambda_{\frac{3(g(2x)-f(2x))}{8}-\frac{g(-2x)-f(-2x)}{8}}\left(\frac{\alpha u}{2}t\right)\\
&\geq \tau_M\left\{\Lambda_{\frac{3(g(2x)-f(2x))}{8}}\left(\frac{3\alpha ut}{8}\right), \Lambda_{\frac{g(-2x)-f(-2x)}{8}}\left(\frac{\alpha ut}{8}\right)\right\}\\
&\geq \tau_M\left\{\Lambda_{g(2x)-f(2x)}(\alpha ut), \Lambda_{g(-2x)-f(-2x)}(\alpha ut)\right\}\\
&\geq \tau_M\left\{\Lambda'_{\varphi(2x,2x,2x,-2x)}(\alpha t), \Lambda'_{\varphi(-2x,-2x,-2x,2x)}(\alpha t)\right\}\\
&\geq M(x,t)
\end{aligned}$$

for all $x \in X\backslash\{0\}$, which implies that

$$d(Jf, Jg) \leq \frac{\alpha}{2}d(f,g).$$

That is, $J$ is a strictly contractive self-mapping of $S$ with the Lipschitz constant $0 < \frac{\alpha}{2} < 1$. It is clear that

$$f(x) - Jf(x) = \frac{3Df(x,x,x,-x)}{16} - \frac{Df(-x,-x,-x,x)}{16}$$

for all $x \in X\backslash\{0\}$. Moreover, by (3.1), we see that

$$\begin{aligned}
\Lambda_{f(x)-Jf(x)}\left(\frac{t}{4}\right) &= \Lambda_{\frac{3Df(x,x,x,-x)}{16}-\frac{Df(-x,-x,-x,x)}{16}}\left(\frac{t}{4}\right)\\
&\geq \tau_M\left\{\Lambda_{\frac{3Df(x,x,x,-x)}{16}}\left(\frac{3t}{16}\right), \Lambda_{\frac{Df(-x,-x,-x,x)}{16}}\left(\frac{t}{16}\right)\right\}\\
&\geq \tau_M\left\{\Lambda_{Df(x,x,x,-x)}(t), \Lambda_{Df(-x,-x,-x,x)}(t)\right\}\\
&\geq \tau_M\left\{\Lambda'_{\varphi(x,x,x,-x)}(t), \Lambda'_{\varphi(-x,-x,-x,x)}(t)\right\}
\end{aligned}$$

for all $x \in X\backslash\{0\}$. It means that $d(f, Jf) \leq \frac{1}{4} < \infty$ by the definition of $d$. Therefore according to Theorem 3.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : X \to Y$ of $J$ in the set $T = \{g \in S | d(f,g) < \infty\}$, which is represented by

$$F(x) := \lim_{n\to\infty}\left(\frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}}\right)$$

for all $x \in X$. Since

$$d(f,F) \leq \frac{1}{1-\frac{\alpha}{2}}d(f,Jf) \leq \frac{1}{2(2-\alpha)}$$

the inequality (3.2) holds. Next we will show that $F$ is a quadratic-additive mapping. Let $x, y, z, w \in X$. Then by (RN3) we have

$$\begin{aligned}
\Lambda_{DF(x,y,z,w)}(t) \geq \tau_M \Bigg\{ & \Lambda_{(F-J^nf)(x+y+z+w)}\left(\frac{t}{30}\right), \Lambda_{2(F-J^nf)(x)}\left(\frac{t}{15}\right), \\
& \Lambda_{2(F-J^nf)(y)}\left(\frac{t}{15}\right), \Lambda_{2(F-J^nf)(z)}\left(\frac{t}{15}\right), \\
& \Lambda_{2(F-J^nf)(w)}\left(\frac{t}{15}\right), \Lambda_{(F-J^nf)(x+y)}\left(\frac{t}{30}\right), \\
& \Lambda_{(F-J^nf)(x+z)}\left(\frac{t}{30}\right), \Lambda_{(F-J^nf)(x+w)}\left(\frac{t}{30}\right), \\
& \Lambda_{(F-J^nf)(y+z)}\left(\frac{t}{30}\right), \Lambda_{(F-J^nf)(y+w)}\left(\frac{t}{30}\right), \\
& \Lambda_{(F-J^nf)(z+w)}\left(\frac{t}{30}\right), \Lambda_{DJ^nf(x,y,z,w)}\left(\frac{t}{2}\right) \Bigg\} \qquad (3.4)
\end{aligned}$$

for all $x, y, z, w \in X\backslash\{0\}$ and $n \in \mathbb{N}$. The first eleven terms on the right hand side of the above inequality tend to 1 as $n \to \infty$ by the definition of $F$. Now consider that

$$\begin{aligned}
& \Lambda_{DJ^nf(x,y,z,w)}\left(\frac{t}{2}\right) \\
& \geq \tau_M \Bigg\{ \Lambda_{\frac{Df(2^nx,2^ny,2^nz,2^nw)}{2\cdot 4^n}}\left(\frac{t}{8}\right), \Lambda_{\frac{Df(-2^nx,-2^ny,-2^nz,-2^nw)}{2\cdot 4^n}}\left(\frac{t}{8}\right), \\
& \qquad \Lambda_{\frac{Df(2^nx,2^ny,2^nz,2^nw)}{2\cdot 2^n}}\left(\frac{t}{8}\right), \Lambda_{\frac{Df(-2^nx,-2^ny,-2^nz,-2^nw)}{2\cdot 2^n}}\left(\frac{t}{8}\right) \Bigg\} \\
& \geq \tau_M \Bigg\{ \Lambda_{Df(2^nx,2^ny,2^nz,2^nw)}\left(\frac{4^nt}{4}\right), \Lambda_{Df(-2^nx,-2^ny,-2^nz,-2^nw)}\left(\frac{4^nt}{4}\right), \\
& \qquad \Lambda_{Df(2^nx,2^ny,2^nz,2^nw)}\left(\frac{2^nt}{4}\right), \Lambda_{Df(-2^nx,-2^ny,-2^nz,-2^nw)}\left(\frac{2^nt}{4}\right) \Bigg\} \\
& \geq \tau_M \Bigg\{ \Lambda'_{\varphi(x,y,z,w)}\left(\frac{4^nt}{4\alpha^n}\right), \Lambda'_{\varphi(-x,-y,-z,-w)}\left(\frac{4^nt}{4\alpha^n}\right), \\
& \qquad \Lambda'_{\varphi(x,y,z,w)}\left(\frac{2^nt}{4\alpha^n}\right), \Lambda'_{\varphi(-x,-y,-z,-w)}\left(\frac{2^nt}{4\alpha^n}\right) \Bigg\}
\end{aligned}$$

which tends to 1 as $n \to \infty$ by (RN3) and $\frac{2}{\alpha} > 1$ for all $x, y, z, w \in X\backslash\{0\}$. Therefore it follows from (3.4) that

$$\Lambda_{DF(x,y,z,w)}(t) = 1$$

for each $x, y, z, w \in X \backslash \{0\}$ and $t > 0$. By (RN1) and Lemma 3.2, this means that $DF(x, y, z, w) = 0$ for all $x, y, z, w \in X$.

**Case 2.** Assume that $\varphi$ satisfies the condition (ii). It follows from (3.1), (3.3), (RN2), and (RN3) that

$$\begin{aligned}
&\Lambda_{f(0)}(5t)\\
&\geq \tau_M \Big\{ \Lambda_{\frac{1}{3}Df(2^n x,2^n x,2^n x,2^n x)}(t), \Lambda_{\frac{1}{3}Df(-2^n x,-2^n x,-2^n x,-2^n x)}(t),\\
&\Lambda_{\frac{1}{3}Df(2^{n+1}x,2^{n+1}x,-2^{n+1}x,-2^{n+1}x)}(t), \Lambda_{\frac{2}{3}Df(2^n x,2^n x,-2^n x,-2^n x)}(2t) \Big\}\\
&\geq \tau_M \Big\{ \Lambda'_{\alpha^n \varphi(x,x,x,x)}(3t), \Lambda'_{\alpha^n \varphi(-x,-x,-x,-x)}(3t),\\
&\Lambda'_{\alpha^n \varphi(2x,2x,-2x,-2x}(3t), \Lambda'_{\alpha^n \varphi(x,x,-x,-x)}(3t) \Big\}\\
&\geq \tau_M \Big\{ \Lambda'_{\varphi(x,x,x,x)}\left(\frac{3t}{\alpha^n}\right), \Lambda'_{\varphi(-x,-x,-x,-x)}\left(\frac{3t}{\alpha^n}\right),\\
&\Lambda'_{\varphi(2x,2x,-2x,-2x}\left(\frac{3t}{\alpha^n}\right), \Lambda'_{\varphi(x,x,-x,-x)}\left(\frac{3t}{\alpha^n}\right) \Big\}
\end{aligned}$$

for a fixed $x \in X \backslash \{0\}$, $t > 0$, and for all $n \in \mathbb{N}$. Since all terms on the last side of the above inequality tend to 1 as $n \to \infty$, we have $f(0) = 0$ by (RN1). The rest proof of this case is same as that of Case 1. In particular, assume that $\Lambda'_{\varphi(x,y,z,w)}$ is continuous in $x, y, z, w$. If $m, a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4$ are any fixed integers with $a_1, a_2, a_3, a_4 \neq 0$, then we have

$$\begin{aligned}
&\lim_{n\to\infty} \Lambda'_{\varphi((2^n a_1+b_1)x,(2^n a_2+b_2)y,(2^n a_3+b_3)z,(2^n a_4+b_4)w)}(t)\\
&\quad \geq \lim_{n\to\infty} \Lambda'_{\varphi\left(\left(a_1+\frac{b_1}{2^n}\right)x,\left(a_2+\frac{b_2}{2^n}\right)y,\left(a_3+\frac{b_3}{2^n}\right)z,\left(a_4+\frac{b_4}{2^n}\right)w\right)}\left(\frac{t}{\alpha^n}\right)\\
&\quad \geq \lim_{n\to\infty} \Lambda'_{\varphi\left(\left(a_1+\frac{b_1}{2^n}\right)x,\left(a_2+\frac{b_2}{2^n}\right)y,\left(a_3+\frac{b_3}{2^n}\right)z,\left(a_4+\frac{b_4}{2^n}\right)w\right)}(mt)\\
&\quad = \Lambda'_{\varphi(a_1 x,a_2 y,a_3 z,a_4 w)}(mt)
\end{aligned}$$

for all $x, y, z, w \in X \backslash \{0\}$ and $t > 0$. Since $m$ is arbitrary, we have

$$\begin{aligned}
&\lim_{n\to\infty} \Lambda'_{\varphi((2^n a_1+b_1)x,(2^n a_2+b_2)y,(2^n a_3+b_3)z,(2^n a_4+b_4)w)}(t)\\
&\quad \geq \lim_{m\to\infty} \Lambda'_{\varphi(a_1 x,a_2 y,a_3 z,a_4 w)}(mt) = 1
\end{aligned}$$

for all $x, y, z, w \in X\backslash\{0\}$ and $t > 0$. From these, we get the inequality

$$\begin{aligned}
&\Lambda_{3(F(x)-f(x))}(13t)\\
&\geq \lim_{n\to\infty} \tau_M\big\{\Lambda_{(Df-DF)((2^n+1)x,-2^nx,-2^nx,-2^nx)}(t),\\
&\qquad \Lambda_{(F-f)((-2^{n+1}+1)x)}(t), \Lambda_{3(f-F)(-2^{n+1}x)}(3t),\\
&\qquad \Lambda_{6(F-f)(-2^nx)}(6t), \Lambda_{2(F-f)((2^n+1)x)}(2t)\big\}\\
&\geq \lim_{n\to\infty} \tau_M\big\{\Lambda'_{\varphi((2^n+1)x,-2^nx,-2^nx,-2^nx)}(t),\\
&\qquad M((1-2^{n+1})x, 2(2-\alpha)t), M\left(2^{n+1}x, 2(2-\alpha)t\right),\\
&\qquad M((2^n+1)x, 2(2-\alpha)t), M\left(2^nx, 2(2-\alpha)t\right)\big\}\\
&= 1
\end{aligned}$$

for all $x \in X \setminus \{0\}$. From the above equality and the fact $f(0) = 0 = F(0)$, we obtain $f \equiv F$.

**Case 3.** Assume that $\varphi$ satisfies the condition (iii). One can show that $f(0) = 0$ by the same method used in Case 1. Let the set $(S, d)$ be as in the proof of Case 1. Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) := g\left(\frac{x}{2}\right) - g\left(-\frac{x}{2}\right) + 2\left(g\left(\frac{x}{2}\right) + g\left(-\frac{x}{2}\right)\right)$$

for all $g \in S$ and $x \in V$. Notice that

$$J^n g(x) = 2^{n-1}\left(g\left(\frac{x}{2^n}\right) - g\left(-\frac{x}{2^n}\right)\right) + \frac{4^n}{2}\left(g\left(\frac{x}{2^n}\right) + g\left(-\frac{x}{2^n}\right)\right)$$

for all $x \in X$. Let $f, g \in S$ and let $u \in [0, \infty]$ be an arbitrary constant with $d(g, f) \leq u$. From the definition of $d$, (RN2), and (RN3), we have

$$\begin{aligned}
&\Lambda_{Jg(x)-Jf(x)}\left(\frac{4u}{\alpha}t\right) = \Lambda_{3(g(\frac{x}{2})-f(\frac{x}{2}))+g(-\frac{x}{2})-f(-\frac{x}{2})}\left(\frac{4u}{\alpha}t\right)\\
&\geq \tau_M\left\{\Lambda_{3(g(\frac{x}{2})-f(\frac{x}{2}))}\left(\frac{3u}{\alpha}t\right), \Lambda_{g(-\frac{x}{2})-f(-\frac{x}{2})}\left(\frac{u}{\alpha}t\right)\right\}\\
&\geq \tau_M\left\{\Lambda_{g(\frac{x}{2})-f(\frac{x}{2})}\left(\frac{u}{\alpha}t\right), \Lambda_{g(-\frac{x}{2})-f(-\frac{x}{2})}\left(\frac{u}{\alpha}t\right)\right\}\\
&\geq \tau_M\left\{\Lambda'_{\varphi(\frac{x}{2},\frac{x}{2},\frac{x}{2},-\frac{x}{2})}\left(\frac{t}{\alpha}\right), \Lambda'_{\varphi(-\frac{x}{2},-\frac{x}{2},-\frac{x}{2},\frac{x}{2})}\left(\frac{t}{\alpha}\right)\right\}\\
&\geq M(x, t)
\end{aligned}$$

for all $x \in X\backslash\{0\}$, which implies that

$$d(Jf, Jg) \leq \frac{4}{\alpha} d(f, g).$$

That is, $J$ is a strictly contractive self-mapping of $S$ with the Lipschitz constant $0 < \frac{4}{\alpha} < 1$. Moreover, by (3.1), we see that

$$\begin{aligned}\Lambda_{f(x)-Jf(x)}\left(\frac{t}{2\alpha}\right) &= \Lambda_{-\frac{1}{2}Df(\frac{x}{2},\frac{x}{2},\frac{x}{2},-\frac{x}{2})}\left(\frac{t}{2\alpha}\right)\\ &= \Lambda_{Df(\frac{x}{2},\frac{x}{2},\frac{x}{2},-\frac{x}{2})}\left(\frac{t}{\alpha}\right)\\ &\geq \Lambda'_{\varphi(\frac{x}{2},\frac{x}{2},\frac{x}{2},-\frac{x}{2})}\left(\frac{t}{\alpha}\right)\\ &\geq \Lambda'_{\varphi(x,x,x,-x)}(t)\end{aligned}$$

for all $x \in X\backslash\{0\}$. It means that $d(f, Jf) \leq \frac{1}{2\alpha} < \infty$ by the definition of $d$. Therefore according to Theorem 3.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : X \to Y$ of $J$ in the set $T = \{g \in S | d(f, g) < \infty\}$, which is represented by

$$F(x) := \lim_{n\to\infty}\left(2^{n-1}\left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right) + \frac{4^n}{2}\left(f\left(\frac{x}{2^n}\right) + f\left(-\frac{x}{2^n}\right)\right)\right)$$

for all $x \in X$. Since

$$d(f, F) \leq \frac{1}{1-\frac{4}{\alpha}}d(f, Jf) \leq \frac{1}{2(\alpha-4)}$$

the inequality (3.2) holds. Next we will show that $F$ is quadratic-additive. Let $x, y, z, w \in X$. Then by (RN3) we have the inequality (3.4) for all $x, y, z, w \in X\backslash\{0\}$ and $n \in \mathbb{N}$. The first eleven terms on the right hand side of the inequality (3.4) tend to 1 as $n \to \infty$ by the definition of $F$. Now consider that

$$\begin{aligned}&\Lambda_{DJ^n f(x,y,z,w)}\left(\frac{t}{2}\right)\\ &\quad\geq \tau_M\left\{\Lambda_{2^{2n-1}Df(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n},\frac{w}{2^n})}\left(\frac{t}{8}\right), \Lambda_{2^{2n-1}Df(\frac{-x}{2^n},\frac{-y}{2^n},\frac{-z}{2^n},\frac{-w}{2^n})}\left(\frac{t}{8}\right),\right.\\ &\qquad\left.\Lambda_{2^{n-1}Df(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n},\frac{w}{2^n})}\left(\frac{t}{8}\right), \Lambda_{-2^{n-1}Df(\frac{-x}{2^n},\frac{-y}{2^n},\frac{-z}{2^n},\frac{-w}{2^n})}\left(\frac{t}{8}\right)\right\}\\ &\quad\geq \tau_M\left\{\Lambda'_{\varphi(x,y,z,w)}\left(\frac{\alpha^n t}{4^{n+1}}\right), \Lambda'_{\varphi(-x,-y,-z,-w)}\left(\frac{\alpha^n t}{4^{n+1}}\right),\right.\\ &\qquad\left.\Lambda'_{\varphi(x,y,z,w)}\left(\frac{\alpha^n t}{2^{n+2}}\right), \Lambda'_{\varphi(-x,-y,-z,-w)}\left(\frac{\alpha^n t}{2^{n+2}}\right)\right\}\end{aligned}$$

which tends to 1 as $n \to \infty$ by (RN3) for all $x, y, z, w \in X\backslash\{0\}$. Therefore it follows from (3.4) that

$$\Lambda_{DF(x,y,z,w)}(t) = 1$$

for each $x, y, z, w \in X\backslash\{0\}$ and $t > 0$. By (RN1) and Lemma 3.2, this means that $DF(x, y, z, w) = 0$ for all $x, y, z, w \in X$. It completes the proof of Theorem 3.3. □

Now we will establish the stability for the functional equation (1.3) in random normed spaces.

THEOREM 3.4. *Let $X$, $(Z, \Lambda', \tau_M)$, $(Y, \Lambda, \tau_M)$, $\varphi$ and $M(x, t)$ be as in Theorem 3.3. If $f : X \to Y$ is a mapping such that*

$$\Lambda_{D'f(x,y,z,w)}(t) \geq \Lambda'_{\varphi(x,y,z,w)}(t) \tag{3.5}$$

*for all $x, y, z, w \in X\backslash\{0\}$ and $t > 0$, then there exists a unique general quadratic mapping $F : X \to Y$ satisfying (3.2) for all $x \in X\backslash\{0\}$ and $t > 0$.*

*Proof.* Let $\tilde{f} = f - f(0)$. Then by (3.5) we have

$$\Lambda_{D\tilde{f}(x,y,z,w)}(t) = \Lambda_{D'f(x,y,z,w)}(t) \geq \Lambda'_{\varphi(x,y,z,w)}(t)$$

for all $x, y, z, w \in X\backslash\{0\}$ and $t > 0$ with $\tilde{f}(0) = 0$. By Theorem 3.3, there exists a unique mapping $F' : X \to Y$ satisfying (3.2) for $\tilde{f}$ and $DF'(x, y, z, w) = 0$. Put $F = F' + f(0)$, then we easily show that $D'F(x, y, z, w) = 0$ and $F$ satisfying (3.2) for $f$. □

Now we have the generalized Hyers-Ulam stability of the quadratic-additive functional equation (1.2) in the framework of normed spaces. Let $\Lambda_x(t) = \frac{t}{t+\|x\|}$. Then $(X, \Lambda, \tau_M)$ is an induced random normed space, which leads us to get the following result.

COROLLARY 3.5. *Let $X$ be a linear space and $Y$ a complete normed-space. And let $f : X \to Y$ be a mapping for which there is $\varphi : (X\backslash\{0\})^4 \to [0, \infty)$ such that*

$$\|Df(x, y, z, w)\| \leq \varphi(x, y, z, w)$$

*for all $x, y, z, w \in X\backslash\{0\}$. If, for all $x, y, z, w \in X\backslash\{0\}$, $\varphi$ satisfies one of the following conditions:*
(i) $\alpha\varphi(x, y, z, w) \geq \varphi(2x, 2y, 2z, 2w) \geq \alpha'\varphi(x, y, z, w)$ *for some* $1 < \alpha' \leq \alpha < 2$,
(ii) $\alpha\varphi(x, y, z, w) \geq \varphi(2x, 2y, 2z, 2w)$ *for some* $0 < \alpha < 1$,
(iii) $\varphi(2x, 2y, 2z, 2w) \geq \alpha\varphi(x, y, z, w)$ *for some* $4 < \alpha$,

*then there exists a unique quadratic-additive mapping $F : X \to Y$ such that*

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{\Phi(x)}{2(2-\alpha)} & \text{if } \varphi \text{ satisfies (i) or (ii)}, \\ \frac{\Phi(x)}{2(\alpha-4)} & \text{if } \varphi \text{ satisfies (iii)} \end{cases}$$

*for all $x \in X$, where $\Phi(x)$ is defined by*

$$\Phi(x) \quad = \max\Big\{\varphi(x, x, x, -x), \varphi(-x, -x, -x, x)\Big\}.$$

*Moreover, if $\varphi$ is continuous under the condition* (ii)*, then $f$ is a quadratic-additive mapping.*

Now we have the Hyers-Ulam-Rassias stability of the quadratic-additive functional equation (1.2) in the framework of normed spaces.

Corollary 3.6. *Let $X$ be a normed space, $p \in (-\infty, 0) \cup (0, 1) \cup (2, \infty)$ and $Y$ a complete normed-space. If $f : X \to Y$ is a mapping such that*

$$\|Df(x, y, z, w)\| \leq \|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p$$

*for all $x, y, z, w \in X\backslash\{0\}$, then there exists a unique quadratic-additive mapping $F : X \to Y$ such that*

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{2\|x\|^p}{2-2^p} & \text{if } 0 < p < 1, \\ \frac{2\|x\|^p}{2^p-4} & \text{if } p > 2 \end{cases}$$

*for all $x \in X\backslash\{0\}$ and $f$ is itself a quadratic-additive mapping if $p < 0$.*

*Proof.* If we denote by $\varphi(x, y, z, w) = \|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p$, then the induced random normed space $(X, \Lambda_x, \tau_M)$ holds the conditions of Theorem 3.3 with $\alpha = 2^p$. □

Corollary 3.7. *Let $X$, $Y$, and $\varphi$ be as in Corollary 3.5. If $f : X \to Y$ is a mapping such that*

$$\|D'f(x, y, z, w)\| \leq \varphi(x, y, z, w)$$

*for all $x, y, z, w \in X\backslash\{0\}$, then there exists a unique general quadratic mapping $F : X \to Y$ satisfying* (3.6).

Now we have the Hyers-Ulam-Rassias stability of the general quadratic functional equation (1.3).

Corollary 3.8. *Let $X$ be a normed space, $p \in (-\infty, 0) \cup (0, 1) \cup (2, \infty)$ and $Y$ a complete normed-space. If $f : X \to Y$ is a mapping such that*

$$\|D'f(x, y, z, w)\| \leq \|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p$$

*for all $x, y, z, w \in X\backslash\{0\}$, then there exists a unique general-quadratic mapping $F : X \to Y$ satisfying (3.7) if $p(0,1) \cup (2,\infty)$ and $f$ is itself a general quadratic mapping if $p < 0$.*

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